

On variational eigenvalue approximation of semidefinite operators

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Abstract

Eigenvalue problems for semidefinite operators with infinite dimensional kernels appear for instance in electromagnetics. Variational discretizations with edge elements have long been analyzed in terms of a discrete compactness property. As an alternative, we show here how the abstract theory can be developed in terms of a geometric property called the vanishing gap condition. This condition is shown to be equivalent to eigenvalue convergence and intermediate between two different discrete variants of Friedrichs estimates. Next we turn to a more practical means of checking these properties. We introduce a notion of compatible operator and show how the previous conditions are equivalent to the existence of such operators with various convergence properties. In particular the vanishing gap condition is shown to be equivalent to the existence of compatible operators satisfying an Aubin-Nitsche estimate. Finally we give examples demonstrating that the implications not shown to be equivalences, indeed are not.

MSC classes : 65J10, 65N25, 65N30.

1 Introduction

The basic eigenvalue problem in electromagnetics reads:

$$\operatorname{curl} \mu^{-1} \operatorname{curl} u = \omega^2 \epsilon u, \quad (1.1)$$

for some vector field u satisfying boundary conditions in a bounded domain in Euclidean space. The matrix coefficients μ and ϵ characterize electromagnetic properties of the medium. The eigenvalue ω^2 is expressed here in terms of the angular frequency ω . The operator on the left is semidefinite with an infinite dimensional kernel. The variational approximation of such eigenvalue problems reads: find u and ω such that for all u' there holds:

$$\int \mu^{-1} \operatorname{curl} u \cdot \operatorname{curl} u' = \omega^2 \int \epsilon u \cdot u'. \quad (1.2)$$

In Galerkin discretizations, u is in some finite dimensional space X_n and the above equation should hold for all u' in X_n . One has an integer parameter n and looks at convergence properties as n increases.

The above eigenvalue problem is of the following general form: find $u \in X$ and $\lambda \in \mathbb{R}$ such that for all $u' \in X$:

$$a(u, u') = \lambda \langle u, u' \rangle, \quad (1.3)$$

where on the right we have the scalar product $\langle \cdot, \cdot \rangle$ of a Hilbert space O (typically an L^2 space) and on the left a symmetric semipositive bilinear form a on a Hilbert space X . In our setting, X will be dense and continuously embedded in O , but not necessarily compactly so, so that the standard theory [3] does not directly apply. The important hypothesis is instead that the O -orthogonal complement of the kernel of a in X is compactly embedded in O . In the context of electromagnetics, such compactness results are included in the discussion of [25].

The most successful Galerkin spaces for (1.2) are those of [36] and the convergence of discrete eigenvalues has been shown to follow essentially from a *discrete compactness* (DC) property of these spaces, as defined in [32]. This notion has been related to that of collectively compact operators [35]. Under some assumptions, eigenvalue convergence is actually equivalent to discrete compactness [14]. Necessary and sufficient conditions for eigenvalue convergence are also obtained in [6] via mixed formulations. Natural looking finite element spaces, which satisfy standard inf-sup conditions, equivalent to a *discrete Friedrichs* estimate (DF), but nevertheless yielding spurious eigenvalues, have been exhibited [7]. For reviews see [29, 34, 5].

Eigenpair convergence has long been expressed in terms of gaps between discrete and continuous eigenspaces. In [17] the analysis of some surface integral operators was based upon another type of gap, which, transposed to the above problem, concerns the distance from discrete divergence free vector fields to truly divergence free ones. In other words, one considers the gap, in its unsymmetrized form, between the discrete and continuous spaces spanned by the eigenvectors attached to non-zero eigenvalues, from the former to the latter. The general framework was further developed in [11], to the effect that discrete inf-sup conditions for non-coercive operators naturally followed from a *vanishing gap* (VG) condition. A variant can be found in [16]. In [10] this theory was applied to the source problem for (1.2) in anisotropic media. Contrary to DC, VG has a clear visual interpretation, expressing a geometric property of abstract discrete Hodge decompositions.

That VG implies DC is immediate. The converse was noted in [13]. Thus VG is, via the above cited results, equivalent to eigenvalue convergence. Moreover DC implies DF, though [7] shows that the converse is not true. However we shall show that VG follows from an optimal version of DF, referred to as ODF henceforth. While unimportant for eigenvalue convergence, the ODF, in the form of a negative norm estimate, was used in [18] (under the name uniform norm equivalence) to prove a discrete div-curl lemma. It was also remarked that a local version of the ODF is implied by the discrete div-curl lemma. A common underlying assumption is that we have approximating discrete kernels (ADK). In section §3 we detail the relationships between ADK, DF, DC (several variants), VG and ODF. We also show how to deduce eigenvalue convergence from VG. While VG uses the norm of X to compute the gap, we relate ODF to a gap property in O . The results on ODF are new. Most of the other results should be considered known in principle, but the ordering and brevity of the arguments might be of interest. A summary is provided in diagram (3.46).

The main tool used to prove DC and error estimates for finite element spaces is the construction of interpolation operators. The role of commuting diagrams they satisfy has been highlighted [4]. However the interpolators deduced from the canonical degrees of freedom of edge elements are not even defined on H^1 for domains in \mathbb{R}^3 . This has lead to great many technical hurdles, involving delicate regularity estimates. Commuting projectors that are *tame*, in the sense of being uniformly

bounded $L^2 \rightarrow L^2$, were proposed in [38]. Another construction combining standard interpolation with a smoothing operator, obtained by cut-off and convolution on reference macro-elements, was introduced in [19]. As pointed out in [1], for flat domains, quasi-uniform grids and natural boundary conditions, one can simplify the construction to use only smoothing by convolution on the entire physical domain. In [23] these restrictions were overcome by the introduction of a space dependent smoother. Such operators yield eigenvalue convergence quite easily, as well as ODF. It should also be remarked that eigenvalue convergence for the discretization of the Hodge-Laplacian by Whitney forms had been obtained much earlier in [27]. For a review of the connection between finite elements and Hodge theory, see [1, 2].

Reciprocally one would like to know if DF, VG or ODF imply the existence of commuting projections with enough boundedness, since this will indicate possible strategies for convergence proofs in more concrete settings. For instance, in §4 of [6], it is shown that eigenvalue convergence for mixed formulations is equivalent to the existence of a Fortin operator, converging in a certain operator norm. In §3.3 of [2] it is shown, in the context of Hilbert complexes, that DF is equivalent to the existence of commuting projections that are uniformly bounded in energy norm. We introduce here a notion of *compatible operator* (CO) which contains both Fortin operators and commuting projections. With this notion at hand, DF is equivalent to the existence of energy bounded COs, VG is equivalent to the existence of COs satisfying an Aubin-Nitsche estimate, and ODF is equivalent to the existence of tame COs. The equivalences of DF, VG and ODF to various properties of compatible operators are detailed in §4.

In §5 we give some complementary results. §5.1 shows how tame compatible operators appear in the context of differential complexes. §5.2 studies what happens when the bilinear forms are replaced by equivalent ones, extending a result proved in [14]. §5.3 considers the optimality of proved implications. We show how to construct approximating subspaces satisfying ADK but not DF. We also show how one can construct spaces satisfying DF but not DC, and spaces satisfying DC but not ODF. Thus all the major proved implications that were not proved to be equivalences, are proved not to be. These results are new in this abstract form, but as already mentioned, [7] gives a concrete counter-example of spaces satisfying DC but not DF.

2 Setting

2.1 Continuous spaces and operators

Let O be an infinite dimensional separable real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. Let X be a dense subspace which is itself a Hilbert space with continuous inclusion $X \rightarrow O$. We suppose that we have a continuous symmetric bilinear form a on X satisfying:

$$\forall u \in X \quad a(u, u) \geq 0. \quad (2.1)$$

Moreover we suppose that the bilinear form $\langle \cdot, \cdot \rangle + a$ is coercive on X so that we may take it to be its scalar product. The corresponding norm on X is denoted $\|\cdot\|$, whereas the natural one on O is denoted $|\cdot|$. Thus:

$$|u|^2 = \langle u, u \rangle, \quad (2.2)$$

$$\|u\|^2 = \langle u, u \rangle + a(u, u). \quad (2.3)$$

Define:

$$W = \{u \in X : \forall u' \in X \quad a(u, u') = 0\}, \quad (2.4)$$

$$V = \{u \in X : \forall w \in W \quad \langle u, w \rangle = 0\}, \quad (2.5)$$

so that we have a direct sum decomposition into closed subspaces:

$$X = V \oplus W, \quad (2.6)$$

which are orthogonal with respect to both $\langle \cdot, \cdot \rangle$ and a .

Since a is semi-definite we have:

$$W = \{u \in X : a(u, u) = 0\}. \quad (2.7)$$

We do *not* require W to be finite-dimensional. Remark that W is closed also in O . Let \overline{V} be the closure of V in O and remark:

$$\overline{V} = \{u \in O : \forall w \in W \quad \langle u, w \rangle = 0\}, \quad (2.8)$$

so that we have an orthogonal splitting of O :

$$O = \overline{V} \oplus W. \quad (2.9)$$

We also have:

$$V = \overline{V} \cap X. \quad (2.10)$$

We let P be the projector in O with range \overline{V} and kernel W . It is orthogonal in O . It maps V to V so it may also be regarded as a continuous projector in X . Since the splitting (2.6) is orthogonal with respect to the scalar product on X defined by (2.3), P is an orthogonal projector also in X .

We suppose that the injection $V \rightarrow O$ is compact (when V inherits the norm topology of X). It implies the Friedrich inequality, namely that there is $C > 0$ such that:

$$\forall v \in V \quad |v|^2 \leq Ca(v, v). \quad (2.11)$$

In particular, on V , a is a scalar product whose associated norm is equivalent to the one inherited from X , defined by (2.3). In X , P can be characterized by the property that for $u \in X$, $Pu \in V$ solves:

$$a(Pu, v) = a(u, v), \quad (2.12)$$

for all $v \in V$. This holds then for all $v \in X$. We will also frequently use the identity, for $u, v \in X$:

$$a(Pu, Pv) = a(u, v). \quad (2.13)$$

Remark 1. Let S be a bounded contractible Lipschitz domain in \mathbb{R}^3 with outward pointing normal ν on ∂S . Equation (1.2) leads to the following functional frameworks, depending on which boundary conditions are used. For simplicity, we take ϵ and μ to be scalar Lipschitz functions on S that are bounded below, above zero.

• Define:

$$O = L^2(S) \otimes \mathbb{R}^3, \quad (2.14)$$

$$X = \{u \in O : \operatorname{curl} u \in O\}, \quad (2.15)$$

$$a(u, u') = \int \mu^{-1} \operatorname{curl} u \cdot \operatorname{curl} u', \quad (2.16)$$

$$\langle u, u' \rangle = \int \epsilon u \cdot u'. \quad (2.17)$$

Then we have:

$$W = \text{grad } H^1(S), \quad (2.18)$$

$$\overline{V} = \{u \in O : \text{div } \epsilon u = 0 \text{ and } u \cdot \nu = 0\}. \quad (2.19)$$

- As a variant, replace in the above definitions:

$$X = \{u \in O : \text{curl } u \in O \text{ and } u \times \nu = 0\}. \quad (2.20)$$

Then we get:

$$W = \text{grad } H_0^1(S), \quad (2.21)$$

$$\overline{V} = \{u \in O : \text{div } \epsilon u = 0\}. \quad (2.22)$$

In both examples, compactness of $V \rightarrow O$ is guaranteed by results in [25]. If the topology of S is more complicated, W can contain, in addition to gradients, a non-trivial albeit finite dimensional, space of harmonic vectorfields. The characterization of \overline{V} should then be modified accordingly.

For the case of more general coefficients ϵ and μ , see §5.2.

The dual of X with O as pivot space is denoted X' . The duality pairing on $X' \times X$ is thus denoted $\langle \cdot, \cdot \rangle$. We let V' denote the subspace of X' consisting of elements u such that $\langle u, v \rangle = 0$ for all $v \in W$.

Let $K : X' \rightarrow X$ be the continuous operator defined as follows. For all $u \in X'$, $Ku \in V$ satisfies:

$$\forall v \in V \quad a(Ku, v) = \langle u, v \rangle. \quad (2.23)$$

If $u \in V'$, equation (2.23) holds for all $v \in X$. One sees that K is identically 0 on V' and induces an isomorphism $V' \rightarrow V$.

Since $V \rightarrow O$ is compact, $K : X' \rightarrow O$ is compact. As an operator $O \rightarrow O$, K is selfadjoint. We deduce that K is compact $O \rightarrow X$ and a fortiori $O \rightarrow O$. We are interested in the following eigenvalue problem. Find $u \in X$ and $\lambda \in \mathbb{R}$ such that:

$$\forall u' \in X \quad a(u, u') = \lambda \langle u, u' \rangle. \quad (2.24)$$

When $W \neq 0$ it is the eigenspace associated with the eigenvalue 0. A positive λ is an eigenvalue of a iff $1/\lambda$ is a positive eigenvalue of K ; the eigenspaces are the same. The non-zero eigenvalues of a form an increasing positive sequence diverging to infinity, each eigenspace being a finite dimensional subspace of V . The direct sum of these eigenspaces is dense in V .

2.2 Discretization

Given the above setting, consider a sequence (X_n) of finite dimensional subspaces of X . We will later formulate conditions to the effect that elements u of X can be approximated by sequences (u_n) such that $u_n \in X_n$. In the mean-time we state some algebraic definitions.

We consider the following discrete eigenproblems. Find $u \in X_n$ and $\lambda \in \mathbb{R}$ such that:

$$\forall u' \in X_n \quad a(u, u') = \lambda \langle u, u' \rangle. \quad (2.25)$$

We decompose:

$$W_n = \{u \in X_n : \forall u' \in X_n \quad a(u, u') = 0\} = X_n \cap W, \quad (2.26)$$

$$V_n = \{u \in X_n : \forall w \in W_n \quad \langle u, w \rangle = 0\}, \quad (2.27)$$

so that:

$$X_n = V_n \oplus W_n. \quad (2.28)$$

Thus $W_n \subseteq W$ but it's crucial, for the points we want to make, that V_n need not be a subspace of V .

Notice that W_n is the 0 eigenspace and that for non-zero $v \in V_n$ we have:

$$a(v, v) > 0. \quad (2.29)$$

Define K_n as follows. For all $u \in X'$, $K_n u \in V_n$ satisfies:

$$\forall v \in V_n \quad a(K_n u, v) = \langle u, v \rangle. \quad (2.30)$$

Define also $P_n : X \rightarrow V_n$ by, for all $u \in X$, $P_n u \in V_n$ satisfies:

$$\forall v \in V_n \quad a(P_n u, v) = a(u, v), \quad (2.31)$$

which then holds for all $v \in X_n$.

For $u \in V'$ we have, for all $v \in V_n$:

$$a(P_n K u, v) = a(K u, v) = \langle u, v \rangle = a(K_n u, v). \quad (2.32)$$

So that:

$$K_n u = P_n K u. \quad (2.33)$$

If, on the other hand $u \in W$, $K u = 0$ so $P_n K u = 0$, but $K_n u = 0$ iff u is O -orthogonal to V_n . If $V_n \not\subseteq V$ there are elements in W not orthogonal to V_n .¹

The non-zero discrete eigenvalues of a are the inverses of the non-zero eigenvalues of K_n .

A minimal assumption for reasonable convergence properties is:

AS We say that (X_n) are Approximating Subspaces of X if:

$$\forall u \in X \quad \lim_{n \rightarrow \infty} \inf_{u' \in X_n} \|u - u'\| = 0. \quad (2.34)$$

Proposition 1. *If AS holds we also have:*

$$\forall u \in O \quad \lim_{n \rightarrow \infty} \inf_{u' \in X_n} |u - u'| = 0, \quad (2.35)$$

$$\forall v \in \overline{V} \quad \lim_{n \rightarrow \infty} \inf_{v' \in V_n} |v - v'| = 0, \quad (2.36)$$

$$\forall v \in V \quad \lim_{n \rightarrow \infty} \inf_{v' \in V_n} \|v - v'\| = 0. \quad (2.37)$$

Proof. (i) The first property holds by density of X in O .

(ii) If we denote by Q_n the O -orthogonal projection onto W_n and $v \in \overline{V}$ is approximated in the O norm by a sequence $u_n \in X_n$ we have:

$$u_n - Q_n u_n \in V_n, \quad (2.38)$$

¹The (crucial) point that $P_n K$ and K_n need not coincide on W was overlooked in [20].

and, since Q_nv is orthogonal to both v and V_n :

$$|v - (u_n - Q_n u_n)| \leq |(v - u_n) - Q_n(v - u_n)| \leq |v - u_n| \rightarrow 0. \quad (2.39)$$

(iii) Define Q_n as above. In X it is a projection with norm 1. Pick $v \in V$ and choose a sequence $u_n \in X_n$ converging to v in X . Remark that Q_nv is orthogonal to both v and V_n , also in X . We get:

$$\|v - (u_n - Q_n u_n)\| \leq \|(v - u_n) - Q_n(v - u_n)\| \leq 2\|v - u_n\| \rightarrow 0. \quad (2.40)$$

This concludes the proof. \square

Statements of the form “there exists $C > 0$ such that for all n , certain quantities f_n and g_n satisfy $f_n \leq Cg_n$ ”, will be abbreviated $f_n \preccurlyeq g_n$. Thus C might depend on the choice of sequence (X_n) but not on n .

3 Friedrichs, gaps and eigenvalue convergence

In this section, and for the rest of this paper, we assume that the spaces (X_n) have been chosen so that AS holds.

Recall that V_n is in general not a subspace of V . Below we will relate convergence of variational discretizations, to how well elements of V_n can be approximated by those of V . More generally one is interested in what properties of elements of V , the elements of V_n share.

For non-zero subspaces U and U' of X one defines the gap $\delta(U, U')$ from U to U' by:

$$\delta(U, U') = \sup_{u \in U} \inf_{u' \in U'} \|u - u'\| / \|u\|. \quad (3.1)$$

In what follows gaps will be computed with respect to the norm of X , unless otherwise specified.

In the above setting we define:

Definition 1. ADK We say that we have Approximating Discrete Kernels if:

$$\forall w \in W \quad \lim_{n \rightarrow \infty} \inf_{w' \in W_n} |w - w'| = 0. \quad (3.2)$$

DF Discrete Friedrichs holds if there is $C > 0$ such that:

$$\forall n \quad \forall v \in V_n \quad |v|^2 \leq Ca(v, v). \quad (3.3)$$

DC Discrete Compactness holds if, from any subsequence of elements $v_n \in V_n$ which is bounded in X , one can extract a subsequence converging strongly in O , to an element of V .

VG We say that Vanishing Gap holds if:

$$\delta(V_n, V) \rightarrow 0 \text{ when } n \rightarrow \infty. \quad (3.4)$$

ODF We say that the Optimal Discrete Friedrichs inequality holds if for some $C > 0$ we have:

$$\forall n \quad \forall v \in V_n \quad |v| \leq C|Pv|. \quad (3.5)$$

This section discusses first the relations between these conditions, and second their relation to the convergence of K_n to K in various senses. The first three conditions are studied in particular in [32, 14], see also [5] §19. There are many possible variants of DC and we discuss several below. For an often overlooked subtlety in the definition of DC, see Remark 2. VG appeared in [17, 11, 10]. ODF is a condition we introduce here, as a counterpart to boundedness in O of certain operators (mimicking so called commuting projections and Fortin operators) that will be discussed later, in §4. Notice that ODF implies DF, due to (2.11) and (2.13). A summary of the proved implications is provided at the end of this section, in diagram (3.46).

Even ADK, which we will see is the weakest of the above hypotheses, is quite restrictive, but arises naturally in discretizations of complexes of Hilbert spaces by subcomplexes as will be explained in §5.1. For the examples of Remark 1 the most important finite element spaces are those of [36]. We will see that they have all the above properties, of which ODF is the strongest.

Strictness of proved implications will be discussed in §5.3 below.

3.1 Internal relations

We first study the implications between these conditions.

Lemma 2. *Suppose ADK does not hold. Then there is a subsequence of elements $v_n \in V_n$ such that:*

- v_n converges weakly in X to a non-zero element of W .
- Pv_n converges strongly to 0 in X .

Proof. For any closed subspace Y of X , let $Q[Y]$ be the X -orthogonal projection onto Y defined by (2.3). Suppose ADK is not satisfied. Choose $w \in W$, $\epsilon > 0$ and an infinite subset N of \mathbb{N} such that for all $n \in N$:

$$|w - Q[W_n]w| \geq \epsilon. \quad (3.6)$$

By AS we have:

$$Q[V_n]w + Q[W_n]w = Q[X_n]w \rightarrow w \text{ in } X. \quad (3.7)$$

Since $Q[V_n]w$ is bounded in X , we may suppose in addition that it converges weakly in X to some w' . Since $(w - Q[W_n]w)$ then also converges weakly to w' , we must have $w' \in W$.

If $w' = 0$, $Q[W_n]w$ converges weakly to w and we can write:

$$\|w\|^2 \leq \liminf \|Q[W_n]w\|^2, \quad (3.8)$$

$$= \liminf (\|w\|^2 - \|w - Q[W_n]w\|^2), \quad (3.9)$$

$$\leq \|w\|^2 - \epsilon^2. \quad (3.10)$$

This is impossible. So $w' \neq 0$.

We have, by (3.7):

$$a(Q[V_n]w, Q[V_n]w) = a(Q[X_n]w, Q[X_n]w) \rightarrow a(w, w) = 0, \quad (3.11)$$

Hence, using (2.13), $PQ[V_n]w$ converges strongly in X to 0.

The subsequence $v_n = Q[V_n]w$ satisfies the stated conditions. \square

Proposition 3. *The following are equivalent:*

- *ADK.*
- *For any subsequence $v_n \in V_n$ that converges weakly in O to some $v \in O$, we have $v \in \overline{V}$.*
- *For any subsequence $v_n \in V_n$ that converges weakly in X to some $v \in X$, we have $v \in V$.*

Proof. (i) Suppose ADK holds. Let $v_n \in V_n$ converge weakly in O to v . Pick $w \in W$ and choose $w_n \in W_n$ converging, in the O -norm, to w . Then we have:

$$\langle v, w \rangle = \lim_{n \rightarrow \infty} \langle v_n, w_n \rangle = 0, \quad (3.12)$$

so that $v \in \overline{V}$.

- (ii) The third statement follows from the second by (2.10).
- (iii) The third statement implies the first by Lemma 2. □

The following corresponds to Proposition 2.21 in [14].

Proposition 4. *DF implies ADK.*

Proof. Suppose ADK does not hold. Let v_n be a subsequence as in Lemma 2. Recalling (2.13), we get that:

$$a(v_n, v_n) = a(Pv_n, Pv_n) \rightarrow 0, \quad (3.13)$$

but v_n cannot converge strongly to 0 in O . Hence DF is not satisfied. □

Remark 2. So far the finite element literature has been rather cavalier about taking *sub*-sequences from the outset in the formulation of DC, probably because it is rarely made clear what the set of indexes is, in the first place. See Remark 19.3 in [5]. Without this precision one cannot hope to deduce eigenvalue convergence, since one cannot rule out that good Galerkin spaces are interspaced with bad ones.

In the formulation of DC some authors, including [14] and [5], prefer not to impose that the limit is in V . Compactness is also frequently used in the form that a weak convergence implies a strong one. For completeness we state the following two alternatives to DC:

DC' We say DC' holds if, from any subsequence of elements $v_n \in V_n$ which is bounded in X , one can extract a subsequence converging strongly in O .

DC'' We say DC'' holds if, for any sequence of elements $v_n \in V_n$ which converges weakly in X , the convergence is strong in O .

Notice that in the last variant, we use sequences, not subsequences – though we could have done that as well. These conditions are related as follows.

Proposition 5. *The following are equivalent:*

- *DC.*
- *DC' and ADK.*

• *DC'' and ADK.*

Proof. (i) Suppose DC holds. DC' trivially holds, so we focus on ADK. Consider a subsequence $v_n \in V_n$ that converges weakly in X to some $v \in X$. Extract from it one that converges strongly in O to an element in V . Since this limit must be v , $v \in V$. This proves ADK by Proposition 3.

(ii) Let a sequence $v_n \in V_n$ converge weakly in X to say v . If it does not converge strongly in O to v choose an $\epsilon > 0$ and a subsequence for which $|v_n - v| \geq \epsilon$. No subsequence of this subsequence can converge strongly in O , since the limit would have to be v . Hence DC' implies DC''.

(iii) Suppose DC'' and ADK hold. Consider a subsequence $v_n \in V_n$ bounded in X . Extract from it one that converges weakly in X to say v . By ADK, $v \in V$. For the indexes n not pertaining to this subsequence, insert the best approximation of v in V_n . Using Proposition 1 we now have a sequence converging weakly in X to v , so by DC'' it converges strongly in O .

This proves DC. \square

Remark 3. Neither DC' nor DC'' implies ADK. Consider the case where W is finite dimensional but non zero. Since both W and V are compactly injected into O , the injection $X \rightarrow O$ is compact, so that DC' and DC'' are automatically satisfied. But one can construct subspaces X_n satisfying AS, whose intersection with W is zero, so that ADK is not satisfied.

Corresponding to the well known fact that DC implies DF, we state:

Proposition 6. *VG implies DF.*

Proof. We have, for $v \in V_n$:

$$\|v - Pv\| \leq \delta(V_n, V)\|v\|, \quad (3.14)$$

which gives:

$$(1 - \delta(V_n, V))\|v\| \leq \|Pv\|. \quad (3.15)$$

For, say, $\delta(V_n, V) \leq 1/2$ we then get:

$$|v| \leq \|v\| \leq 2\|Pv\| \preccurlyeq a(Pv, Pv)^{1/2} = a(v, v)^{1/2}, \quad (3.16)$$

which completes the proof. \square

The following is an abstract variant of Proposition 7.13 in [13].

Proposition 7. *DC is equivalent to VG.*

Proof. (i) Suppose VG holds. Suppose a subsequence of elements $v_n \in V_n$ is bounded in X . Then $Pv_n \in V$ is bounded in X , so it has a subsequence converging strongly in O and weakly in X . By (2.10) the limit, call it v , is in V . We have:

$$|v_n - Pv_n| \leq \delta(V_n, V)\|v_n\| \rightarrow 0, \quad (3.17)$$

so that v_n converges in O to v . Hence DC holds.

(ii) Suppose that DC holds but not VG. We get a contradiction as follows. Choose $\epsilon > 0$ and a subsequence of elements $v_n \in V_n$ which is bounded in X but such that, for all n pertaining to the subsequence:

$$\|v_n - Pv_n\| \geq \epsilon. \quad (3.18)$$

Extract a subsequence that converges strongly in O to some $v \in V$. Then (Pv_n) also converges to v in O . We can now write:

$$|v_n - v| \geq |v_n - Pv_n| - |Pv_n - v| = \|v_n - Pv_n\| - |Pv_n - v|, \quad (3.19)$$

which is eventually bounded below by $\epsilon/2$, contradicting the convergence of (v_n) . \square

As already indicated, ODF implies DF, but actually something stronger is true:

Proposition 8. *ODF implies DC.*

Proof. Suppose ODF holds. Consider a subsequence $v_n \in V_n$, bounded in X . Then Pv_n is bounded in V , so we may find a subsequence converging strongly in O to some $v \in V$. Choose $u_n \in V_n$ converging strongly to v in O (Proposition 1). Then Pu_n converges to v in O . Using ODF we deduce:

$$|v_n - u_n| \leq |Pv_n - Pu_n|, \quad (3.20)$$

$$\leq |v - Pv_n| + |v - Pu_n| \rightarrow 0. \quad (3.21)$$

Therefore v_n converges to v strongly in O . \square

One can also consider gaps defined by the norm of O rather than X . Explicitly, for subspaces U and U' of O we define:

$$\delta_O(U, U') = \sup_{u \in U} \inf_{u' \in U'} |u - u'|/|u|. \quad (3.22)$$

Proposition 9. *ODF is equivalent to the condition: there exists $\delta < 1$ such that for all n , $\delta_O(V_n, \overline{V}) \leq \delta$.*

Proof. Remark first that for all n and all non-zero $v \in V_n$, $|v - Pv| < |v|$, so by compactness of the unit sphere of V_n , $\delta_O(V_n, \overline{V}) < 1$.

(i) If the stated condition did not hold we would have a subsequence of elements $v_n \in V_n$ such that $|v_n| = 1$ and $|v_n - Pv_n|$ converges to 1. Then we have:

$$|Pv_n|/|v_n| = (1 - |v_n - Pv_n|^2)^{1/2} \rightarrow 0. \quad (3.23)$$

Hence ODF does not hold.

(ii) If on the other hand the stated condition holds we have, for all n and $v \in V_n$:

$$|Pv|^2 = |v|^2 - |v - Pv|^2 \geq (1 - \delta^2)|v|^2, \quad (3.24)$$

and this gives ODF. \square

Remark 4. In the context of curl problems with natural boundary conditions in convex domains (see Remark 1), the ODF has the following equivalent formulation in terms of a negative norm:

$$\forall n \forall v \in V_n \quad \|v\|_{L^2(S)} \leq C \|\operatorname{curl} v\|_{H^{-1}(S)}. \quad (3.25)$$

Here, V_n is the subspace of vectorfields in X_n that are orthogonal to discrete gradients (of functions that can be non-zero on the boundary). In this form the condition was discussed in connection with a discrete div curl lemma in [18]. An advantage of the present formulation is to avoid negative norms and boundary conditions.

In [21] it was pointed out that for the discrete div curl lemma to hold, ODF is actually necessary, at least on periodic domains.

Remark 5. An interpretation of results in [21] (§3), is that for some standard spaces satisfying ODF (the edge elements of [36] for instance) we do *not* have that $\delta_O(V_n, \overline{V})$ converges to 0.

3.2 Convergence of operators

We now explore the relation of the conditions DF and VG to the convergence of the operators K_n .

We first state, concerning P_n :

Proposition 10. *The following are equivalent:*

- DF.
- For any $u \in X$, $P_n u$ converges to Pu in X .
- The P_n are uniformly bounded $X \rightarrow X$.
- The P_n are uniformly bounded $V \rightarrow O$.

Proof. (i) Suppose DF holds, and pick $u \in X$. Choose $v_n \in V_n$ converging to Pu in X (Proposition 1). We have:

$$\|P_n u - v_n\|^2 \preceq a(P_n u - v_n, P_n u - v_n) = a(Pu - v_n, P_n u - v_n) \quad (3.26)$$

$$\preceq \|Pu - v_n\| \|P_n u - v_n\|, \quad (3.27)$$

which gives (quasi-optimal) convergence of $P_n u$ to Pu .

(ii) Pointwise convergence of P_n implies uniform boundedness $X \rightarrow X$ by the principle. Uniform boundedness $X \rightarrow X$ implies uniform boundedness $V \rightarrow O$, of course.

(iii) Suppose DF does not hold. Pick a subsequence $v_n \in V_n$ such that $|v_n| = 1$ but $a(v_n, v_n) \rightarrow 0$. Since $a(Pv_n, Pv_n) = a(v_n, v_n)$, the sequence (Pv_n) converges to 0 in V . On the other hand $P_n Pv_n = v_n$, whose O -norm is 1.

This precludes uniform boundedness of $P_n : V \rightarrow O$. \square

We state similar results for K_n , the only subtlety being the behaviour of K_n on W . This particular point will be further expanded upon in Remark 6.

Proposition 11. *The following are equivalent:*

- DF.
- For any $u \in X'$, $K_n u$ converges to Ku in X .
- The K_n are uniformly bounded $X' \rightarrow X$.
- The K_n are uniformly bounded $O \rightarrow O$.

Proof. (i) Suppose DF holds.

Pick $u \in V'$. Then $K_n u = P_n K u$ and $K u \in X$, so by the preceding proposition $K_n u \rightarrow K u$ in X .

Pick $u \in W$. By ADK (Proposition 4) we can choose a sequence $u_n \in W_n$ converging to u in W_n . We have:

$$\|K_n u\|^2 \preceq a(K_n u, K_n u) = \langle u - u_n, K_n u \rangle \quad (3.28)$$

$$\preceq |u - u_n| \|K_n u\|. \quad (3.29)$$

Hence $K_n u \rightarrow 0 = K u$ in X .

This proves the second statement.

(ii) The second statement implies the third, which in turn implies the last.

(iv) When the last condition holds, the largest eigenvalue of K_n is uniformly bounded, which implies DF. \square

The above propositions show, as is well known, that DF is the key estimate for the convergence of the Galerkin method (2.30) for source problems. Turning to eigenvalue approximation, DF simply says that the smallest non-zero discrete eigenvalue of a is bounded below, above zero. It is well known, since [7], that DF by itself is not enough to guarantee eigenvalue convergence. This has been the main reason for considering extra conditions, like DC [32].

To study the convergence of the spectral attributes (such as eigenvalues) of K_n , to those of K , a number of conditions have been used [28, 31, 15], in quite general settings that include unbounded operators. The most convenient sufficient condition for us, appears to be:

NC We say Norm Convergence holds if $\|K - K_n\|_{O \rightarrow O} \rightarrow 0$.

In [37], see [3] §7, it is shown directly that if NC holds, then for any non-zero eigenvalue λ of K , any small enough disk around λ in \mathbb{C} , the eigenvalues of K_n inside it converge to λ , with corresponding convergence of eigenspaces in the sense of gaps. A key intermediate step is the norm convergence of the spectral projections associated with the disc, written as contour integrals. The argument easily extends to any contour in \mathbb{C} that does not meet the spectrum of K or enclose 0.

In [7] §5 a reciprocal is proved, for positive and injective K : a convergence of eigenvalues and spaces in the sense of gaps, implies NC.

Arguably DF is not necessary for eigenvalue convergence of the discretizations (2.25). Indeed one can imagine a scenario where some discrete nonzero eigenvalues of a cluster around 0, arbitrarily close as $n \rightarrow \infty$, with corresponding eigenspaces close to W , in the sense of gaps. This scenario need not be catastrophic in practice, for instance to someone who needs to identify resonances of an electromagnetic device. In the case where W is finite dimensional this is even more patent: any approximating subspaces X_n will yield a reasonable notion of eigenvalue convergence, even those that do not contain W . For such discretizations, K_n is not even uniformly bounded in O .

However our present theory is motivated by the situation where DF, and hence ADK, holds. *We then take it for granted that eigenvalue and eigenspace convergence for (2.25), properly defined in terms of gaps, is equivalent to NC.*

In [14] Definition 6.1, a notion of “spurious free” approximation is introduced. It is quite easily seen to imply DF. In Theorem 6.10 of that paper, it is shown that spurious free approximation is equivalent to DC (and AS). They use results of [26] which concern a weaker variant of NC, applicable when K is bounded but not necessarily compact.

We now prove the analogue result for us, namely that NC is equivalent to VG.

Lemma 12. *We have:*

$$\delta(V_n, V) \leq \|K_n\|_{W \rightarrow X}. \quad (3.30)$$

Proof. For any $v \in V_n$ we have:

$$|v - Pv|^2 = \langle v - Pv, v \rangle, \quad (3.31)$$

$$= a(K_n(v - Pv), v), \quad (3.32)$$

$$\leq \|K_n(v - Pv)\| \|v\|, \quad (3.33)$$

$$\leq \|K_n\|_{W \rightarrow X} |v - Pv| \|v\|. \quad (3.34)$$

We deduce:

$$\|v - Pv\| \leq \|K_n\|_{W \rightarrow X} \|v\|. \quad (3.35)$$

This proves the claim. \square

Proposition 13. *If DF holds then:*

$$\|K - K_n\|_{\overline{V} \rightarrow X} \rightarrow 0, \quad (3.36)$$

and:

$$\|K_n\|_{W \rightarrow X} \simeq \delta(V_n, V), \quad (3.37)$$

Proof. By DF, the P_n are projectors $X \rightarrow V_n$ which converge pointwise to P (Proposition 10). Since $K : O \rightarrow X$ is compact, Lemma 23 in the appendix then gives $\|K - P_n K\|_{O \rightarrow X} \rightarrow 0$, which gives (3.36) since K_n and $P_n K$ coincide on \overline{V} , see (2.33).

Pick $u \in W$. We have:

$$a(K_n u, K_n u) = \langle u, K_n u \rangle, \quad (3.38)$$

$$= \langle u, (I - P) K_n u \rangle, \quad (3.39)$$

$$\leq |u| \delta(V_n, V) \|K_n u\|. \quad (3.40)$$

Since we also have, by DF:

$$\|K_n u\|^2 \preceq a(K_n u, K_n u), \quad (3.41)$$

we get:

$$\|K_n u\| \preceq \delta(V_n, V) |u|. \quad (3.42)$$

Combined with (3.30) this gives (3.37). \square

Proposition 14. *The following are equivalent:*

- VG.
- $\|K - K_n\|_{O \rightarrow X}$ tends to 0.
- NC.

Proof. (i) If VG holds then DF holds. Applying Proposition 13 shows that $\|K - K_n\|_{O \rightarrow X}$ tends to 0.

(ii) We have:

$$\|K - K_n\|_{O \rightarrow O} \leq \|K - K_n\|_{O \rightarrow X}. \quad (3.43)$$

(iii) Suppose that $\|K - K_n\|_{O \rightarrow O} \rightarrow 0$.

For $u \in W$ we have:

$$a(K_n u, K_n u) = \langle u, K_n u \rangle \leq |u| |K_n u|. \quad (3.44)$$

Therefore we have:

$$\|K_n\|_{W \rightarrow X} \rightarrow 0. \quad (3.45)$$

By Lemma 12, VG holds. \square

Remark 6. When DF holds, Proposition 13 shows that the question of eigenvalue convergence can be stated entirely in terms of the behaviour of K_n on W , as a distinction between pointwise and uniform convergence of K_n . For any u in W , $K_n u \rightarrow 0$ in O (Proposition 11), but eigenvalue convergence holds iff $\|K_n\|_{W \rightarrow O}$ tends to 0.

3.3 Summary

The following diagram represents the main implications proved in this section. The numbers refer to propositions.

$$\begin{array}{ccccc}
 ODF & \xleftrightarrow{9} & SG & & \\
 \Downarrow 8 & & & & \\
 DC & \xleftrightarrow{7} & VG & \xleftrightarrow{14} & NC \\
 & & \Downarrow 6 & & \\
 & & DF & \xleftrightarrow{11} & PC \\
 & & \Downarrow 4 & & \\
 & & ADK & &
 \end{array} \tag{3.46}$$

PC stands for Pointwise Convergence of the solution operators K_n , as in the second point of Proposition 11. SG stands for the Small Gap condition for V_n in O , considered in Proposition 9.

4 Compatible operators and convergence criteria

In the finite element setting the most useful convergence results for variational discretizations are derived from projection operators such as those associated with the degrees of freedom. A key part of our investigation is therefore to relate the convergence criteria discussed above to the existence of such operators. For this purpose we have found it useful to introduce the following notion of a compatible operator.

Definition 2. We say that an operator $Q_n : X \rightarrow X_n$ is *compatible* with X_n , if:

- it maps W into W_n ,
- for any $u \in X_n$, $u - Q_n u \in W_n$.

A sequence of operators Q_n is compatible if for each n , Q_n is compatible with X_n .

For instance the P_n defined by (2.31) are compatible operators. They have range V_n and map W to 0. Also, projections onto X_n satisfying commuting diagrams, as will be detailed in §5.1 below, are compatible. They have range X_n and project W onto W_n .

For projections Q_n onto X_n , the second condition in the above definition of compatibility is trivially satisfied.

Remark 7. In the discussion of [5], so-called Fortin operators play a central role (see §14). They are operators $Q_n : X \rightarrow X_n$ such that:

$$\forall u \in X \quad \forall v \in V_n \quad a(u - Q_n u, v) = 0, \tag{4.1}$$

which then holds for all $v \in X_n$. Thus P_n is the only Fortin operator with range V_n . Any Fortin operator is compatible in the above sense but the reciprocal is false : In the context of curl problems a Fortin operator would need to map the curl according to the L^2 projection. In general, commuting projections don't have this property.

To deduce convergence results from compatible operators, norm properties are necessary. We state:

Definition 3. FCO Friedrichs Compatible Operators are compatible sequences of operators which are uniformly bounded $X \rightarrow X$.

ANCO Aubin-Nitsche Compatible Operators are compatible operators Q_n such that $\|\text{id} - Q_n\|_{V \rightarrow O}$ converges to 0. In other words ANCO states that for some $\epsilon_n \rightarrow 0$ we have:

$$\forall v \in V \quad \forall n \quad \|v - Q_n v\| \leq \epsilon_n \|v\|. \quad (4.2)$$

TCO Tame Compatible Operators are compatible operators which are uniformly bounded $O \rightarrow O$. We say that TCO holds whenever such operators exist.

We have already encountered uniform boundedness $X \rightarrow X$, as in FCO, in Proposition 10, where it was related to DF. The kind of norm convergence stipulated by ANCO corresponds to the so-called Fortin property of [6], see [5] §14 (the Fortin operator should converge to the identity in a certain norm). The construction of commuting projections that are tame in the sense of being uniformly bounded $O \rightarrow O$, is a more recent tool for the analysis of mixed finite elements, see §5.1 below for references. They appear to be a convenient, but not necessary, tool for proving eigenvalue convergence.

The following result can be compared with §3.3 in [2], which treats commuting projections for Hilbert complexes.

Proposition 15. *The following are equivalent:*

- *DF.*
- *The projectors P_n are FCO.*
- *There exist FCO.*

Proof. In view of Proposition 10 we only need to show that the last condition implies the first. Suppose that Q_n are FCO. Then for any $u \in V_n$ we have:

$$u - Q_n P u = (u - Q_n u) + Q_n(u - P u) \in W_n, \quad (4.3)$$

so that $u - Q_n P u$ and u are orthogonal. We deduce:

$$\|u\| \leq \|Q_n P u\| \preccurlyeq \|P u\|, \quad (4.4)$$

hence:

$$|u|^2 \leq \|u\|^2 \preccurlyeq a(P u, P u) = a(u, u), \quad (4.5)$$

which is DF. \square

The following result can be compared with §4 in [6] which treats eigenvalue problems in saddlepoint form. That setting is more general than ours, but the analysis relies on additional notions not explicitly introduced here (weak and strong approximability).

Proposition 16. *The following are equivalent:*

- VG .
- The projectors P_n are ANCO.
- There exist ANCO.

Proof. (i) Suppose VG holds. We shall show that P_n are ANCO. For $v \in V$ we write:

$$|v - P_nv| \leq |P_nv - PP_nv| + |v - PP_nv|. \quad (4.6)$$

For the first term on the right hand side we remark that:

$$|P_nv - PP_nv| \leq \delta(V_n, V) \|P_nv\| \preccurlyeq \delta(V_n, V) \|v\|. \quad (4.7)$$

For the second term we use that for all $v' \in \overline{V}$:

$$\langle v - PP_nv, v' \rangle = a(v - PP_nv, Kv'), \quad (4.8)$$

$$= a(v - P_nv, Kv' - P_nKv'), \quad (4.9)$$

$$\leq \|v - P_nv\| \|Kv' - P_nKv'\|, \quad (4.10)$$

$$\preccurlyeq \|v\| \|K - P_nK\|_{O \rightarrow X} |v'|. \quad (4.11)$$

This gives:

$$|v - PP_nv| \preccurlyeq \|K - P_nK\|_{O \rightarrow X} \|v\|. \quad (4.12)$$

From the compactness of $K : O \rightarrow X$ and Lemma 23 in the appendix, one concludes that $\|K - P_nK\|_{O \rightarrow X}$ converges to 0.

Combining the bounds for the two terms in the right hand side of (4.6), we obtain that P_n are ANCO.

(ii) Suppose Q_n are ANCO, so that we have a sequence $\epsilon_n \rightarrow 0$ verifying:

$$\forall v \in V \ \forall n \quad |v - Q_nv| \leq \epsilon_n \|v\|. \quad (4.13)$$

Now for $v \in V_n$ we have:

$$v - Q_nPv = (v - Q_nv) + Q_n(v - Pv) \in W_n, \quad (4.14)$$

and, since both v and Pv are orthogonal to W_n :

$$|v - Pv| \leq |Pv - Q_nPv| \leq \epsilon_n \|Pv\|. \quad (4.15)$$

This gives VG . □

Remark 8. Proposition 13 shows that when DF holds we have:

$$\|K - K_n\|_{O \rightarrow X} \preccurlyeq \max\{\delta(V_n, V), \|K - P_nK\|_{O \rightarrow X}\}. \quad (4.16)$$

Part (ii) of the proof of Proposition 16 gives an estimate for $\delta(V_n, V)$ from any ANCO. On the other hand P_n converges pointwise quasi-optimally to P in X (see Proposition 10 and equation (2.40)), so that the second term is bounded by the approximation rate:

$$\sup_{u \in O} \inf_{u_n \in X_n} \|Ku - u_n\| / |u|. \quad (4.17)$$

Combining these remarks gives convergence *rates* for $\|K - K_n\|_{O \rightarrow X}$.

Remark 9. To establish VG or some related results one can get away with even less. For instance in the finite element literature one has used operators Q_n that are projections onto X_n mapping (a dense subset of) W to W_n , but that are not defined on all of X . The crucial point is that Q_n should be defined on PX_n and that an estimate of the type (4.15) still holds. This is done in particular in Lemma 4.1 of [24], which involves a local finite dimensionality argument.

Proposition 17. *The following are equivalent:*

- ODF.
- There exist TCO.

Proof. (i) Suppose ODF holds. In $V_n \oplus W$ let Q_n be the projection with range V_n and kernel W . For $u \in V_n \oplus W$ we have:

$$P(u - Q_n u) = 0. \quad (4.18)$$

Using ODF we deduce:

$$|Q_n u| \leq |PQ_n u| = |Pu| \leq |u|. \quad (4.19)$$

For any closed subspace Z of O , let $P[Z]$ denote the O -orthogonal projection onto it. Define:

$$R_n = Q_n P[V_n \oplus W]. \quad (4.20)$$

They have range in V_n , map W to 0 and are uniformly bounded in O , so constitute TCOs.

(ii) We reason as in the proof of Proposition 15. Suppose that Q_n are TCOs. Then for any $u \in V_n$ we have:

$$u - Q_n P u = (u - Q_n u) + Q_n(u - P u) \in W_n. \quad (4.21)$$

Hence $u - Q_n P u$ is orthogonal to u in O and we have:

$$|u| \leq |Q_n P u| \leq |P u|. \quad (4.22)$$

This gives ODF. □

Remark 10. Notice that we have proved that if TCO holds, then ODF holds by Proposition 17, so DC holds by Proposition 8, so VG holds by Proposition 7, so the P_n are ANCO by Proposition 16 – however they will in general not be tame. For instance, the variational H^1 projections onto continuous piecewise affine finite element functions are not continuously extendable to L^2 and a fortiori not L^2 stable.

Remark 11. From Lemma 23 it follows that projections that are tame also satisfy the Aubin-Nitsche property, but in general TCOs do not need to be FCOs. On the other hand, tame commuting projections, discussed below, are uniformly bounded in both X and O . Thus the notion of TCO is weaker than that of tame commuting projections.

Remark 12. If ODF holds, one can modify the definition (4.20) to:

$$R_n = Q_n P[V_n \oplus W] + P[W_n]. \quad (4.23)$$

Restricted to V_n and W_n , R_n is the identity, so that now R_n is a projection with range X_n instead of V_n . It is also uniformly bounded $O \rightarrow O$, and maps $W \rightarrow W_n$. Still, boundedness $X \rightarrow X$ does not seem guaranteed and this construction does not appear to provide tame commuting projections in general.

5 Additional remarks

5.1 Hodge Laplacian and commuting diagrams

To study the de Rham complex on a manifold S of dimension n , let the space O^k consist of differential k -forms in $L^2(S)$, and X^k consist of the elements of O^k with exterior derivative in O^{k+1} . We have a complex of Hilbert spaces, linked by the exterior derivative:

$$X^0 \xrightarrow{d} X^1 \xrightarrow{d} \dots \xrightarrow{d} X^n. \quad (5.1)$$

More generally consider a Hilbert complex of spaces X^k ($k \in \mathbb{Z}$). There is a generic notion of Hodge Laplacian at each index k , which can be given a weak formulation involving X^k and X^{k-1} , see [2]. The eigenvalue problem can be decoupled into three problems: one semidefinite eigenvalue problem on X^k , a similar one for X^{k-1} and one concerning harmonic forms. The discretization of these are all analyzed in terms of commuting projections that are tame (uniformly bounded in O^k) in §3.6 of [2].

Tame commuting projections have been constructed in concrete examples [38, 19, 1, 23], corresponding to mixed finite elements for the de Rham complex, in the h -version. It is still an open problem if they exist in the p - and hp -versions of the finite element method. However DC has been verified for important problems in that context [8, 9]. In §5.3 we will check, at an abstract level, that VG does not imply ODF. Therefore, as a tool to study eigenvalue convergence, tame commuting projections are more powerful than what is strictly necessary, to the extent that they might not exist. But recall from Remark 4 that ODF is necessary for some non-linear estimates. Remark 13 gives additional motivation for constructing commuting projections that are stable in weak norms.

We show here how each of the above mentioned semi-definite eigenvalue problems deduced from Hodge Laplacians fits into the framework adopted in this paper, indicating in particular how commuting projections lead to compatible operators.

Suppose that for $k = 0, 1$ we have Hilbert spaces (O^k) and (X^k) which can be arranged as follows:

$$\begin{array}{ccc} O^0 & & O^1 \\ \uparrow & & \uparrow \\ X^0 & \xrightarrow{d} & X^1 \end{array} \quad (5.2)$$

The vertical maps are inclusions and the horizontal map $d : X^0 \rightarrow X^1$ is bounded.

The scalar product on O^k is denoted $\langle \cdot, \cdot \rangle_k$ and we put, for $u, u' \in X^0$:

$$a(u, u') = \langle du, du' \rangle_1. \quad (5.3)$$

We suppose that the triplet (O^0, X^0, a) complies with the setting of §2. In particular X^0 is dense in O^0 and the norm on X^0 can be taken to be defined by:

$$\|u\|^2 = \langle u, u \rangle_0 + \langle du, du \rangle_1. \quad (5.4)$$

We split as before:

$$X^0 = V^0 \oplus W^0. \quad (5.5)$$

We remark that W^0 , the kernel of a , is also the kernel of $d : X^0 \rightarrow X^1$. It is part of the hypotheses that the injection $V^0 \rightarrow O$ must be compact.

We suppose that for $k = 0, 1$ we have a sequence of subspaces (X_n^k) of X^k , such that $d : X_n^0 \rightarrow X_n^1$. We split them as before $X_n^0 = V_n^0 \oplus W_n^0$. One sees that W_n^0 is the kernel of d on X_n^0 .

To check ADK we can do the following. Suppose we have dense subspaces Y^k of X^k , which are also Banach spaces with continuous inclusions, and that we have projections $Q_n^k : Y^k \rightarrow X_n^k$ such that the following diagrams commute:

$$\begin{array}{ccc} Y^0 & \xrightarrow{d} & Y^1 \\ \downarrow Q_n^0 & & \downarrow Q_n^1 \\ X_n^0 & \xrightarrow{d} & X_n^1 \end{array} \quad (5.6)$$

Then Q_n^0 maps $W^0 \cap Y^0$ to W_n^0 . This guarantees ADK to hold, under the mild assumption that $W^0 \cap Y^0$ is dense in W^0 and Q_n^0 uniformly bounded $Y^0 \rightarrow O^0$.

If, in diagram (5.6), $Y^k = X^k$, then Q_n^0 is a compatible operator according to our definition. In particular if it is FCO, ANCO or TCO, corresponding estimates hold, guaranteeing convergence of various variational problems, as detailed in the preceding sections.

In the mixed finite element setting, this reasoning can be applied to the canonical interpolators. The Sobolev injection theorems provide dense Banach spaces Y^k on which the degrees of freedom are well defined, proving ADK. Unfortunately canonical interpolators are not in general well defined on X^k , even less so on O^k . In the above cited references, concerning finite element exterior calculus, tame commuting projections have been constructed by composing canonical interpolators, with smoothing operators that commute with the exterior derivative.

Remark 13. For the de Rham complex, tameness of commuting projections is boundedness with respect to the L^2 norm. Commuting projections that are L^q bounded for other real q , are of interest for non-linear problems. For instance they were constructed and used in [22] (§5.4) to prove discrete translation estimates and Sobolev injections, extending [30] and [12].

5.2 Equivalent forms

For the case of electromagnetics, as in equation (1.2) and Remark 1, it is proved in [14] (Proposition 2.27) that DC holds for electromagnetic coefficients ϵ and μ that are both scalar and equal to 1, if and only if DC holds for coefficients which are 3×3 symmetric matrix fields which are bounded and uniformly positive definite. In this section we give an abstract variant of this result.

Suppose that, in addition to the previous spaces and forms, we have a symmetric bilinear form $\langle \cdot, \cdot \rangle^\sharp$ on O and a symmetric bilinear form a^\sharp on X , which are equivalent to $\langle \cdot, \cdot \rangle$ and a :

$$\forall u \in O \quad \langle u, u \rangle \preccurlyeq \langle u, u \rangle^\sharp \preccurlyeq \langle u, u \rangle, \quad (5.7)$$

$$\forall u \in X \quad a(u, u) \preccurlyeq a^\sharp(u, u) \preccurlyeq a(u, u). \quad (5.8)$$

These bilinear forms can be used to redefine the norm of O as well as that of X . Notice that a and a^\sharp have the same kernel W . Applying the previous constructions

we get new splittings, with obvious definitions:

$$X = V^\sharp \oplus W, \quad (5.9)$$

$$O = \overline{V}^\sharp \oplus W. \quad (5.10)$$

We are interested in the eigenvalue problem these forms define, and its discretization on the same sequence of spaces X_n as before. They get a new splitting:

$$X_n = V_n^\sharp \oplus W_n. \quad (5.11)$$

First we check:

Proposition 18. *The injection $V^\sharp \rightarrow O$ is compact.*

Proof. From (5.10) it follows that the projector P (defined by $\langle \cdot, \cdot \rangle$ and a) induces a bijection $\overline{V}^\sharp \rightarrow \overline{V}$ which is continuous in O . By the open mapping theorem its inverse, denoted J , is also continuous.

Suppose E is a bounded subset of V^\sharp . Then PE is bounded in V , hence relatively compact in \overline{V} . Therefore JPE is relatively compact in \overline{V}^\sharp . But $E = JPE$ so that E is relatively compact in O . \square

Now one can compare the meaning of DF, VG and ODF with respect to $\langle \cdot, \cdot \rangle$ and a , with their corresponding statements for $\langle \cdot, \cdot \rangle^\sharp$ and a^\sharp , which we abbreviate as DF^\sharp , VG^\sharp , ODF^\sharp .

Proposition 19. *We have:*

- DF and DF^\sharp are equivalent.
- VG and VG^\sharp are equivalent.
- ODF and ODF^\sharp are equivalent.

Proof. We use the notion of compatible operator, which is the same for the two choices of bilinear forms.

(i) The first statement follows from Proposition 15 and the third from Proposition 17.

(ii) We now prove that VG implies VG^\sharp , which yields the second statement, since the two choices of bilinear forms play symmetric roles. Define P and P_n by $\langle \cdot, \cdot \rangle$ and a , as before. We also let Q_n denote the projection onto W_n , defined by the bilinear form $\langle \cdot, \cdot \rangle$ and put:

$$R_n = P_n + Q_n. \quad (5.12)$$

Notice that R_n is compatible operator, in fact a projection onto X_n mapping W onto W_n .

For $v \in V^\sharp$ we write:

$$v - R_nv = (I - R_n)Pv + (I - R_n)(I - P)v, \quad (5.13)$$

$$= (I - P_n)Pv + (I - Q_n)(I - P)v. \quad (5.14)$$

The first term on the right hand side is handled by Proposition 16, which yields an estimate (with $\epsilon_n \rightarrow 0$):

$$|(I - P_n)Pv| \leq \epsilon_n \|v\|. \quad (5.15)$$

For the second term, we notice that by Proposition 18, $I - P$ defines a compact operator $V^\sharp \rightarrow O$, with values in W . Recall that VG implies ADK, so that Q_n converges pointwise to the identity on W . We can therefore apply Lemma 23, to get an estimate (with $\epsilon'_n \rightarrow 0$):

$$|(I - Q_n)(I - P)v| \preccurlyeq \epsilon'_n \|v\|. \quad (5.16)$$

Together these estimates prove that R_n is an ANCO with respect to $\langle \cdot, \cdot \rangle^\sharp$ and a^\sharp . Therefore VG $^\sharp$ holds. \square

5.3 Strictness of implications

Remark 14. If W is finite dimensional, ADK implies that for n big enough $W_n = W$ so $V_n \subseteq V$, so that ODF holds.

In the following propositions we suppose that W is infinite-dimensional. We then prove that among the above proved implications, those that were not proved to be equivalences indeed are not.

Proposition 20. *Spaces satisfying AS and ADK can be perturbed along one sequence of directions so as to still satisfy AS and ADK but not DF.*

Proof. Let X_n be spaces satisfying AS and ADK. Pick $v_n \in V_n$ with norm 1 in X but converging weakly to 0 in X . Pick w_n in W , orthogonal to X_n and such that $|w_n| = 1$. Choose a sequence $\epsilon_n > 0$, converging to 0 faster than $|v_n|$. That is, we impose $\epsilon_n = o(|v_n|)$. Define an operator Z_n by:

$$Z_n u = u - \epsilon_n^{-1} \langle u, v_n \rangle / |v_n| w_n. \quad (5.17)$$

It is injective on X_n . Put $\tilde{X}_n = Z_n X_n$. We claim that \tilde{X}_n satisfies AS and ADK but not DF.

Remark that for $u, u' \in X_n$:

$$a(Z_n u, Z_n u') = a(u, u'), \quad (5.18)$$

and that \tilde{X}_n splits according to (2.28) as follows:

$$\tilde{W}_n = W_n \text{ and } \tilde{V}_n = Z_n V_n. \quad (5.19)$$

Then the spaces \tilde{X}_n satisfy AS and ADK. Remark that:

$$a(Z_n v_n, Z_n v_n) = a(v_n, v_n) \preccurlyeq 1 \quad \text{but} \quad |Z_n v_n|^2 = |v_n|^2 (1 + \epsilon_n^{-2}) \rightarrow \infty, \quad (5.20)$$

so that DF does not hold. \square

In [7] finite element spaces satisfying DF but not DC are discussed. Here is an abstract variant.

Proposition 21. *Spaces satisfying AS and DF can be augmented by one vector so as to still satisfy AS and DF but not DC.*

Proof. Suppose spaces X_n satisfy DF. Pick a sequence $v_n \in V$ such that $\|v_n\| = 1$ and v_n is orthogonal to V_n in O and X . Pick also a sequence $w_n \in W$ such that $|w_n| = 1$ and w_n is orthogonal to W_n in O . Put now $u_n = v_n + w_n$ and set:

$$\tilde{X}_n = X_n \oplus \mathbb{R}u_n. \quad (5.21)$$

We claim that \tilde{X}_n satisfies DF but not DC. Indeed \tilde{X}_n splits as follows:

$$\tilde{W}_n = W_n \text{ and } \tilde{V}_n = V_n \oplus \mathbb{R}u_n, \quad (5.22)$$

and DF holds. The sequence (u_n) converges to 0 weakly in X but not strongly in O . \square

The following shows that TCO, while sufficient, is not necessary for eigenvalue convergence.

Proposition 22. *Spaces satisfying AS and VG can be augmented by one vector so as to still satisfy AS and VG but not ODF.*

Proof. Let spaces X_n satisfy VG. Choose v_n in V orthogonal to X_n such that $\|v_n\| = 1$. Then v_n converges weakly to 0 in X and strongly in O . Choose also w_n in W orthogonal to X_n and such that $|w_n| = |v_n|^{1/2}$.

Now put $u_n = v_n + w_n$ and set:

$$\tilde{X}_n = X_n \oplus \mathbb{R}u_n. \quad (5.23)$$

As before, \tilde{X}_n splits according to (2.28) as follows:

$$\tilde{W}_n = W_n \text{ and } \tilde{V}_n = V_n \oplus \mathbb{R}u_n, \quad (5.24)$$

We have:

$$|u_n - Pu_n| = |w_n| \rightarrow 0 \text{ and } \|u_n\| \geq \|v_n\| = 1, \quad (5.25)$$

hence VG still holds.

But we also have:

$$|Pu_n|/|u_n| = |v_n|/(|v_n|^2 + |w_n|^2)^{1/2} = 1/(1 + |v_n|)^{1/2} \rightarrow 0. \quad (5.26)$$

so ODF does not hold. \square

6 Appendix

A variant of the following Lemma was already used in [33]. We temporarily forget about the previous notations.

Lemma 23. *Let X, Y and Z be three Hilbert spaces, $K : X \rightarrow Y$ compact, $A : Y \rightarrow Z$ bounded and $A_n : Y \rightarrow Z$ a uniformly bounded sequence of operators such that:*

$$\forall u \in X \quad \|AKu - A_nKu\|_Z \rightarrow 0. \quad (6.1)$$

Then:

$$\|AK - A_nK\|_{X \rightarrow Z} \rightarrow 0. \quad (6.2)$$

For instance (due to the uniform boundedness principle) this lemma can be applied if (A_n) is a sequence of bounded operators $Y \rightarrow Z$ converging pointwise (in norm) to some A .

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